we have $\cot B = \cot \phi - \cot C \ldots (6)$ and also $\cot B = -\dfrac{\sin(\phi - C)}{\sin\phi \sin C} \ldots (7)$

or $\tan B = \dfrac{\sin\phi \sin C}{\sin(C-\phi)} \ldots (8)$.

As to the practical value of this formula, the less we have to open the book of logarithms the better the formula is, and therefore $\tan \dfrac{B-A}{2} = \dfrac{b-a}{b+a} \cot \dfrac{C}{2}$ is preferable to the above, but as regards the theoretical rank, it is perhaps of the same degree.

It is interesting to notice that an important formula can also be obtained with the same processes from the oblique-angled spherical triangle, (but only for limited cases). Given again a, b, C, by drawing a spherical triangle and lettering it the same as in the above figure,

$\dfrac{\sin a}{\sin b} = \dfrac{\sin A}{\sin B} \ldots (\text{I}) = \dfrac{\sin(B+C)}{\sin B} \ldots (\text{II}).$ $\dfrac{\sin a}{\sin b} = \cos C + \cot B \sin C \ldots$ (III), $\cot B = \dfrac{\sin a - \sin b \cos C}{\sin b \sin C} = \dfrac{\sin a}{\sin b \sin C} - \cot C.$

We might again assume $\dfrac{\sin a}{\sin b \sin C} = \cot\phi$, and proceed as above, or find the tangent or cotangent of the sum of the two unknown angles divided by two.

NON-EUCLIDEAN GEOMETRY, HISTORICAL AND EXPOSITORY.

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CHAPTER FIRST.

EUCLID.

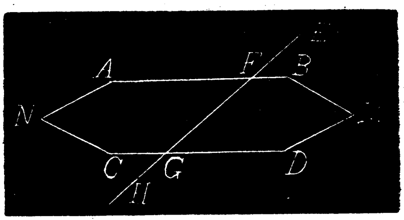
When Alexander the Great was conquering the world, from Macedonia to the Indus, from the Caspian sea to the cataracts of the Nile, founding at least eighteen cities named Alexandria, little did he think that with the only one of these the name now suggests would be connected a man destined to give his name to the universe; for all spaces are now Euclidean or Non-Euclidean.

Euclid, after the death of Alexander, was called by Ptolemy Lagus to open the mathematical school of the first true university, that at Alexandria, and on its teaching so impressed his individuality, that henceforth his name and his immortal Elements stood for his science itself.

Says De Morgan: "As to writing another work on geometry, the middle ages would as soon have thought of composing another New Testament. ....his order of demonstration was thought to be necessary, and founded in the nature of our minds.

The story about Pascal's discovery of geometry in his boyhood (A. D. 1635) contains the statement that he had got 'as far as the 32nd proposition of the first book' before he was detected, the exaggerators (for much exaggerated this very circumstance shows the truth must have been), not having the slightest idea that a new invented system could proceed in any other order than that of Euclid." But even from remote antiquity objection had been made against one point in this imperishable bible of geometry, against Euclid's treatment of parallels.

The great mathematician Ptolemy (died A. D. 168), whose system of astronomy ruled alone until Copernicus, in a work on pure geometry of which Proclus has preserved extracts, discussed the propriety of Euclids famous *Parallel-Postulate*, and proposed to substitute for Euclid's treatment the following: Let the straight $EFGH$ meet the straights AB and CD so as to make the sum of the conjugate angles BFG and FGD equal to two right angles.



It is required to prove that AB and CD are parallel. If possible let them not be parallel, then they will meet when produced say at M (or N). But the angle AFG is the supplement of BFG, and is therefore equal to FGD: similarly the angle FGC is equal to the angle BFG. Hence, the sum of the angles AFG and FGC is equal to two right angles, and the lines BA and DC will therefore if produced meet at N (or M). But two straight lines cannot enclose a space, therefore, AB and CD cannot meet when produced, that is they are parallel.

Inversely, if AB and CD be parallel, then AF and CG are not less parallel than BF and GD; and therefore whatever be the sum of the angles AFG and FGC such must also be the sum of the angles FGD and BFG.

But the sum of the four angles is equal to four right angles, and therefore the sum of the angles BFG and FGD must be equal to two right angles.

Proclus calls this *paralogismos* and *deixeos astheneia;* and Barocius the Venetian translator in 1560, notes it in the margin as *Flagitiosa Ptolemaei ratiocinatio.* "The palpable weakness is, that there is no proof, evidence, or cause of probability assigned, why parallelism should be connected with the angles on one side being together equal to those on the other; the very question in debate being, whether they may not be a little more than two right angles on one side and a little less on the other, and still the straight lines not meet."

The famous Nasir-Eddin (about A. D. 1240), who was born at Tus in Khorassan and says that Euclid was born there also (the Arabs tried to claim oriental origin or education for all the great Greek mathematicians) founds a parallel-theory on two postulates: I. If perpendiculars let fall from one straight onto another make, on the same side, acute angles with the first, these perpendiculars are always shorter toward the acute-angled side, longer toward the other side.

II. Inversely, if the perpendiculars to one straight always grow shorter toward one side, their angles with the other straight are acute on this side.

In a manuscript copy of Euclid in Arabic but in a Persian hand, bought at Ahmedabad, the editor, on the introduction of the Parallel-Postulate, says: "I maintain that the last proposition is not among the universally acknowledged truths, nor anything that is demonstrated in any other part of geometry.

The best way therefore would be that it should be put among the questions instead of the principles; and I shall demonstrate it in a suitable place.

And I lay down for this purpose another proposition, which is, that straight lines in the same plane, if they are subject to an increase of distance on one side, will not be subject to a diminution of distance on that same side, and the contrary; but will cut one another."

Clavius, author of an edition of the Elements: Euclidis elementorum, libri XV.; accessit XVI. de solidorum regularium comparatione, etc. Romae, 1574, announces that "a line every point in which is equally distant from a straight line in the same plane, is a straight line;" upon taking which for granted he infers the properties of parallels. He supports his assumption on the ground that because the acknowledged straight line is one which lies evenly [*ex aequo*] between its extreme points, the other line must do the same, or it would be impossible that it should be everywhere equidistant from the first. He adds: Neque vero cogitatione apprehendi potest aliam lineam praeter rectam, posse habere omnia sua puncta a recta linea, quae in eodem cum illa plano existat, aequaliter distantia. [Clavii Opera. In Euclid. Lib. I. p. 50.]

But it may be remarked, that though the equidistantial be situated *ex aequo* with reference to the primitive straight, we know not whether it posesses this property with reference to itself.

Borelli defines the parallel to a straight by saying that it is the line equidistant from the straight.

In a tract now in the British Museum, printed in 1604, by Dr. Thomas Oliver of Bury, entitled *De rectarum linearum parallelismo et concursu Geometrica*, two demonstrations are proposed; both of them depending on taking for granted, that if a perpendicular of fixed length moves along a straight line, its extremity describes a straight line.

Wolfius, Boscovich, Thomas Simpson, and Bonnycastle define parallels as "straight lines which preserve always the same distance from one another", by distance being understood the length of the perpendicular drawn from a point in one of the straight lines to the other.

But no evidence is adduced that straight lines in any assignable position, *will* always preserve the same distance from one another; nor that the equidistantial is a straight.

Varignon, (1654—1722), the intimate friend of Leibnitz, proposes to define parallels to be "straight lines which are equally inclined to a third straight line," or in other words, make equal corresponding angles. By this he either intends to assume the principal thing at issue, which is whether all straights meet except those making such angles; or he intends to admit none to be parallels except those making equal corresponding angles with some one straight; in which case it must also be assumed, that because they make equal angles with one straight, they shall also do it with any.